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Torsional buckling of a DWCNT embedded on winkler and pasternak foundations using nonlocal theory †

M. Mohammadimehr[1], A. R. Saidi[1], A. Ghorbanpour Arani[2,*], A. Arefmanesh[2] and Q. Han[3]

[1]*Department of Mechanical Engineering, Shahid Bahonar University of Kerman, Kerman, I. R. Iran*
[2]*Department of Mechanical Engineering, Faculty of Engineering, University of Kashan, Kashan, I. R. Iran*
[3]*College of Civil Engineering and Transportation, South China University of Technology, Guangzhou, 510640, P. R. China*

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Abstract

The small-scale effect on the torsional buckling of a double-walled carbon nanotube (DWCNT) embedded on Winkler and Pasternak foundations is investigated in this study using the theory of nonlocal elasticity. The effects of the surrounding elastic medium, such as the spring constant of the Winkler type and the shear constant of the Pasternak type, as well as the van der Waals (vdW) forces between the inner and the outer nanotubes are taken into account. Finally, based on the theory of nonlocal elasticity and by employing the continuum models, an elastic double-shell model is presented for the nonlocal torsional buckling load of a DWCNT. It is seen from the results that the shear constant of the Pasternak type increases the nonlocal critical torsional buckling load, while the difference between the presence and the absence of the shear constant of the Pasternak type becomes large. It is shown that the nonlocal critical buckling load is lower than the local critical buckling load. Moreover, a simplified analysis is carried out to estimate the nonlocal critical torque for the torsional buckling of a DWCNT.

Keywords: DWCNT; Small scale effect; Torsional buckling; VdW force; Winkler and Pasternak foundations

1. Introduction

Iijima first discovered carbon nanotubes (CNTs) in 1991 [1]. Recently, studies have shown that CNTs have excellent mechanical and electronic properties because they are made of a highly ordered sheet of carbon atoms rolled into a tube. Ru [2] and Ranjbartoreh et al. [3] studied the behavior of a double-walled carbon nanotube (DWCNT) that is surrounded by an elastic medium under axial compression. They considered the Winkler model for the surrounding elastic medium on the outer tube and the van der Waals (vdW) forces between two adjacent tubes. Han and Lu [4] investigated the torsional buckling of a DWCNT that is embedded in an elastic medium. They also considered the effects of the surrounding elastic medium and the vdW forces between two adjacent nanotubes in their analysis. Meanwhile, He et al. [5] modeled the vdW forces for axially compressed multi-walled carbon nanotubes (MWCNTs). They obtained explicit formulas for predicting the critical axial strain of a triple-walled CNT using a more refined vdW forces model. Their analysis of a continuum cylindrical shell model of MWCNTs using this refined vdW forces model was carried out to estimate the effects of vdW interaction between the different layers of a CNT and the size of a CNT on the vdW interaction. Han et al. [6] examined the instability of a DWCNT that is embedded in an elastic medium under pure bending by taking into account the vdW forces between the inner and the outer nanotubes. Wang et al. [7] studied the torsional buckling of MWCNTs without consideration of the surrounding elastic medium, while Yang and Wang [8] studied the torsional buckling of MWCNTs without and with consideration of the surrounding elastic medium, respectively. These authors took into account the vdW forces from adjacent nanotubes. Yao and Han [9] investigated the thermal effect on the axially compressed buckling of a MWCNT. They took into account the effects of temperature change, the surrounding elastic medium and the vdW forces between two adjacent layers. Wang et al. [10] conducted an investigation of the elastic buckling of an embedded MWCNT under combined torsion and axial loading. They took into account the radial constraint from the surrounding elastic medium and the vdW forces between two adjacent tube walls. On the other hand, Sohi and Naghdabadi [11] studied the torsional buckling of carbon nanopeapods using a continuum-based multi-layered shell models. These models have taken into

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*Corresponding author. Tel.: +98 913 162 6594, Fax: +98 361 555 9930
E-mail address: aghorban@kashanu.ac.ir

account considered non-bonded vdW interaction between nested fullerenes and the inner-most layer of the host nanotube.

At nano length scales, the effects of size become very important. For this reason, many researchers have addressed the small-scale effect [12-14]. The theory of nonlocal continuum mechanics was first introduced by Eringen in 1983 [15]. He regarded the stress state at a given point as a function of the strain state of all points in the body, while the local continuum mechanics assumed that the stress state at a given point depends uniquely on the strain state at the same point.

Ghorbanpour Arani et al. [16] investigated the transverse vibrations of single-walled carbon nanotube (SWCNT) and DWCNTs under axial load by applying the Euler–Bernoulli and Timoshenko beam models, as well as the Donnell shell model. They concluded that predictions from the Euler–Bernoulli beam model and the Donnell shell model have the lowest and highest accuracies, respectively. In order to predict the vibration behavior of the CNT more accurately, the current classical models were modified using the nonlocal theory. Moreover, they obtained the natural frequencies and amplitude coefficient for the simple supported boundary conditions. Sadeghi et al. [17] investigated the mechanical stability of conductive SWCNTs under applied electric field and compressive loading. They obtained the distribution of electric charges on the nanotube surface by employing a method based on the classical electrostatic theory. Their results showed that in the presence of uniform axial electric fields, the stability of SWCNTs under compressive axial loading increases. Moreover, they studied the effects of CNT geometry on the mechanical stability in electric fields and showed that the increase in the stability of the nanotube depends on the length and diameter. Ghorbanpour Arani et al. [18] investigated the torsional and axially compressed bucklings of an individual MWCNT that has been subjected to internal and external pressures. They also considered the effects in the small scale and the surrounding elastic medium. Their results showed that the internal pressure increased the critical load, while the external pressure tended to decrease it. Ghorbanpour Arani et al. [19] also studied the pure axially compressed buckling and combined the loading effects of a cylindrical shell and an individual SWCNT. They presented the results of the finite element (FE) simulations of SWCNT using the ANSYS software and compared these with the classical (local) and continuum (nonlocal) mechanical theories. Meanwhile, Mohammadimehr et al. [20] developed the Timoshenko beam model for the elastic buckling of DWCNT under axial pressure based on the theory of nonlocal elasticity. Moreover, they obtained the governing equations of equilibrium using the force and moment equilibrium relationships. They compared the Timoshenko beam results with those of the Euler- Bernoulli beam.

Motivated by this concept, this paper investigates the small-scale effect on the torsional buckling of a DWCNT embedded on Winkler and Pasternak foundations. The effects of the surrounding elastic medium, such as the Winkler and the Pasternak models, as well as the vdW forces between the inner and the outer nanotubes, are considered. Finally, a simplified analysis is also presented to estimate the critical torque for torsional buckling of a DWCNT.

2. Nonlocal cylindrical shell model

The nonlocal elasticity model was first presented by Eringen in 1983 [15]. According to this model, the stress at a reference point in the body is dependent not only on the strain state at that point, but also on the strain state at all of the points throughout the body. The constitutive equation of the nonlocal elasticity can be written as follows [15]:

$$(1-(e_0a)^2\,\nabla^2)\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}\,, \tag{1}$$

where C_{ijkl} is the elastic module tensor of the classical isotropic elasticity; and σ_{ij} and ε_{ij} are the stress and strain tensors, respectively. In addition, e_0a is a constant parameter that showing the effect in the small scale.

Consider a circular cylindrical shell with a middle radius r, thickness t, Young's modulus E, and Poisson's ratio μ. The coordinate system is considered as the origin at the middle surface of the shell; x and y denote the axial and the circumferential coordinates of the shell, respectively; and the z direction is the normal coordinate to the medium surface.

The relationships between the strain and the displacement can be written as follows [21]:

$$\begin{aligned}
&\varepsilon_x = u,_x + \frac{1}{2}w,_x{}^2 - zw,_{xx}\\
&\varepsilon_y = v,_y + \frac{w}{r} + \frac{1}{2}w,_y{}^2 - zw,_{yy}\\
&2\varepsilon_{xy} = u,_y + v,_x + w,_x\,w,_y - 2zw,_{xy}
\end{aligned} \tag{2}$$

where, the symbols $\varepsilon_x, \varepsilon_y$, and ε_{xy} denote the strain components at any point through the shell wall thickness; and u, v, and w refer to the displacement components of the middle surface of the shell in the axial, circumferential, and radial directions, respectively.

Using the principle of virtual work and applying Eq. (2), the equations of the equilibrium can be obtained as follows [21]:

$$\begin{aligned}
&N_{x,x} + N_{xy,y} = 0\\
&N_{xy,x} + N_{y,y} = 0\\
&M_{x,xx} + 2M_{xy,xy} + M_{y,yy} - \frac{1}{r}N_y\\
&+N_x\,w,_{xx} + 2N_{xy}\,w,_{xy} + N_y\,w,_{yy} + q(x,y) = 0
\end{aligned} \tag{3}$$

The stress, in terms of the strain according to the Hooke's law via the nonlocal elasticity based on Eq. (1), can be written for the plane stress state as follows:

$$\begin{aligned}
&\sigma_x-(e_0a)^2\nabla^2\sigma_x=\frac{E}{1-\mu^2}(\varepsilon_x+\mu\varepsilon_y)\\
&\sigma_y-(e_0a)^2\nabla^2\sigma_y=\frac{E}{1-\mu^2}(\varepsilon_y+\mu\varepsilon_x)\\
&\sigma_{xy}-(e_0a)^2\nabla^2\sigma_{xy}=\frac{E}{(1+\mu)}\varepsilon_{xy}
\end{aligned}\tag{4}$$

Substituting Eq. (2) into Eq. (4), making them linear and integrating the resulting expressions yield the following resultant forces and moments:

$$\begin{aligned}
&N_x-(e_0a)^2\nabla^2N_x=\frac{C}{r}\left[ru,_x+\mu(rv,_y+w)\right]\\
&N_y-(e_0a)^2\nabla^2N_y=\frac{C}{r}\left[(rv,_y+w)+\mu r u,_x\right]\\
&N_{xy}-(e_0a)^2\nabla^2N_{xy}=C\frac{(1-\mu)}{2}\left[u,_y+v,_x\right]
\end{aligned}\tag{5a}$$

$$\begin{aligned}
&M_x-(e_0a)^2\nabla^2M_x=-D\left[w,_{xx}+\mu w,_{yy}\right]\\
&M_y-(e_0a)^2\nabla^2M_y=-D\left[w,_{yy}+\mu w,_{xx}\right],\\
&M_{xy}-(e_0a)^2\nabla^2M_{xy}=-D(1-\mu)w,_{xy}
\end{aligned}\tag{5b}$$

where, $C=\dfrac{Et}{1-\mu^2}$ and $D=\dfrac{Et^3}{12(1-\mu^2)}$

Substituting Eqs. (5a) and (5b) into Eq. (3), the governing equations of the equilibrium can be obtained as follows:

$$\begin{aligned}
&u,_{xx}+\frac{1-\mu}{2}u,_{yy}+\frac{1+\mu}{2}v,_{xy}+\mu\frac{w,_x}{r}=0\\
&\frac{1+\mu}{2}u,_{xy}+\frac{1-\mu}{2}v,_{xx}+v,_{yy}+\frac{w,_y}{r}=0\\
&D\nabla^4w+\frac{1}{r^2}C(rv,_y+w+\mu r u,_x)\\
&-\left[1-(e_0a)^2\nabla^2\right]\left[N_xw,_{xx}+2N_{xy}w,_{xy}+N_yw,_{yy}\right]\\
&=\left[1-(e_0a)^2\nabla^2\right]q(x,y)
\end{aligned}\tag{6}$$

where $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$, $\nabla^4=\dfrac{\partial^4}{\partial x^4}+2\dfrac{\partial^4}{\partial x^2\partial y^2}+\dfrac{\partial^4}{\partial y^4}$

$q(x,y)$ is the total inward normal pressure and (N_x,N_{xy},N_y) denotes the total membrane forces.

For the post-buckling state, the following relations hold:

$$\begin{aligned}
&N_x=N_{x0}+N_{x1}\\
&N_y=N_{y0}+N_{y1}\\
&N_{xy}=N_{xy0}+N_{xy1}
\end{aligned}\tag{7a}$$

$$\begin{aligned}
&u(x,y)=u_0(x,y)+u_1(x,y)\\
&v(x,y)=v_0(x,y)+v_1(x,y)\\
&w(x,y)=w_0(x,y)+w_1(x,y)\\
&q(x,y)=p_0(x,y)+p(x,y)
\end{aligned}\tag{7b}$$

where the subscript "0" denotes the pre-buckling state, and the subscript "1" refers to infinitesimal increments of the corresponding parameters during the buckling. In Eq. (7b), $p_0(x,y)$ is the normal pressure prior to buckling, and $p(x,y)$ is the additional normal pressure after buckling.

For the pre-buckling state, the governing equations of the equilibrium are as follows:

$$\begin{aligned}
&u_0,_{xx}+\frac{1-\mu}{2}u_0,_{yy}+\frac{1+\mu}{2}v_0,_{xy}+\mu\frac{w_0,_x}{r}=0\\
&\frac{1+\mu}{2}u_0,_{xy}+\frac{1-\mu}{2}v_0,_{xx}+v_0,_{yy}+\frac{w_0,_y}{r}=0\\
&D\nabla^4w_0+\frac{1}{r^2}C(rv_0,_y+w_0+\mu r u_0,_x)\\
&-\left[1-(e_0a)^2\nabla^2\right]\left[N_{x0}w_0,_{xx}+2N_{xy0}w_0,_{xy}+N_{y0}w_0,_{yy}\right]\\
&=\left[1-(e_0a)^2\nabla^2\right]p_0(x,y)
\end{aligned}\tag{8}$$

Substituting Eqs. (7a) and (7b) into Eqs. (6) and applying Eqs. (8), as well as ignoring the rotation angles at the pre-buckling status, $w_0,_x$ and $w_0,_y$, and the high-order small quantities due to the post-buckling deformation $N_{x1}w_1,_{xx}$, $N_{y1}w_1,_{yy}$, and $N_{xy1}w_1,_{xy}$ [9]. As a result, after buckling, the governing equations of the equilibrium can be written as follows:

$$u_1,_{xx}+\frac{1-\mu}{2}u_1,_{yy}+\frac{1+\mu}{2}v_1,_{xy}+\mu\frac{w_1,_x}{r}=0\tag{9a}$$

$$\frac{1+\mu}{2}u_1,_{xy}+\frac{1-\mu}{2}v_1,_{xx}+v_1,_{yy}+\frac{w_1,_y}{r}=0\tag{9b}$$

$$\begin{aligned}
&D\nabla^4w_1+\frac{1}{r^2}C(rv_1,_y+w_1+\mu r u_1,_x)\\
&-\left[1-(e_0a)^2\nabla^2\right]\left[N_{x0}w_1,_{xx}+2N_{xy0}w_1,_{xy}+N_{y0}w_1,_{yy}\right]\\
&=\left[1-(e_0a)^2\nabla^2\right]p(x,y)
\end{aligned}\tag{9c}$$

The steps for the decoupling of Eqs. (9a), (9b) and (9c) are shown in Appendix A. As a result, the governing equations of the equilibrium can be written as:

$$\begin{aligned}
&\nabla^4u_1=-\frac{\mu}{r}w_1,_{xxx}+\frac{1}{r}w_1,_{xyy}\\
&\nabla^4v_1=-\frac{2+\mu}{r}w_1,_{xxy}-\frac{1}{r}w_1,_{yyy}
\end{aligned}\tag{10a}$$

$$\begin{aligned}
&D\nabla^8w_1+\frac{1-\mu^2}{r^2}Cw_1,_{xxxx}\\
&-\left[1-(e_0a)^2\nabla^2\right]\nabla^4\left[N_{x0}w_1,_{xx}+2N_{xy0}w_1,_{xy}+N_{y0}w_1,_{yy}\right]\\
&=\left[1-(e_0a)^2\nabla^2\right]\nabla^4p
\end{aligned}\tag{10b}$$

Replacing $w_1(x,y)$ with $W(x,y)$, Eq. (10b) can be simplified as follows:

$$\begin{aligned}
&D\nabla^8W+\frac{1-\mu^2}{r^2}CW,_{xxxx}\\
&-\left[1-(e_0a)^2\nabla^2\right]\nabla^4\left[N_{x0}W,_{xx}+2N_{xy0}W,_{xy}+N_{y0}W,_{yy}\right]\\
&=\left[1-(e_0a)^2\nabla^2\right]\nabla^4p
\end{aligned}\tag{11}$$

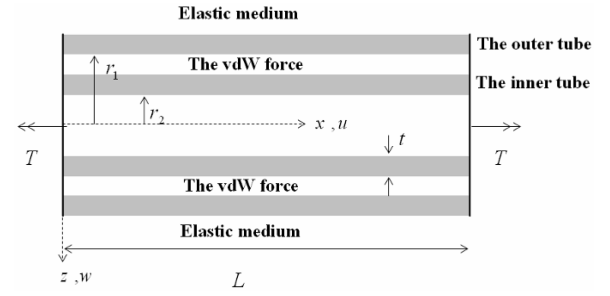

Fig. 1. A DWCNT embedded in an infinite elastic medium with loading configuration.

Eq. (11) can be used to analyze the axial compression, bending, torsion, and radial compression buckling of cylindrical shells surrounded by an elastic medium.

The effects of the surrounding elastic medium, such as the spring constant of the Winkler type and the shear constant of the Pasternak type, on the critical buckling load is considered as the following [22]:

$$F = KW - G\nabla^2 W \ , \tag{12a}$$

where K is the foundation modulus (the spring constant of the Winkler type), and G is the shear modulus of the foundation (the shear constant of the Pasternak type) that are determined by the material properties of the elastic medium and the radius of the outer tube.

The non-dimensional parameters can be defined as follows [22]:

$$\bar{\mu} = \frac{Kr^2(1-\mu^2)}{Et}, \quad \bar{G} = \frac{G(1-\mu^2)}{Et} \tag{12b}$$

where $\bar{\mu}$ is a non-dimensional foundation modulus, and $\bar{G}$ denotes a non-dimensional shear modulus.

Fig. 1 illustrates a DWCNT embedded in an infinite elastic medium with loading configuration. This figure shows a DWCNT of the outer radius r_1, the inner radius r_2, length L, and thickness t.

The vdW force between two carbon atoms can be described by the Lennard-Jones model. For the outer tube, which is denoted by the subscript "1," the normal pressure p_1 can be considered as follows:

$$p_1 = p_1^{\ V} + p_1^{\ W-P} \ , \tag{13}$$

where $p_1^{\ V}$ is the vdW force between the inner and the outer tubes, and $p_1^{\ W-P}$ denotes the interaction pressure due to the elastic medium.

For the inner tube denoted by the subscript "2," the normal pressure p_2 is as follows:

$$p_2 = p_2^{\ V} \tag{14}$$

It is generally known that the interaction forces between the inner and the outer tubes are equal in magnitude and are opposite in sign; therefore, this can be illustrated by the following:

$$p_1^{\ V}(x,y) r_1 = -p_2^{\ V}(x,y) r_2 \ , \tag{15}$$

where r_1 and r_2 are the radii of the outer and the inner tubes, respectively.

The pressure caused by the vdW forces at any point (x,y) on the outer tube is assumed to be a function of the distance $\delta(x,y)$ between the inner and the outer tubes at that point [4]:

$$p_1^{\ V}(x,y) = g[\delta(x,y)] \ , \tag{16}$$

where $g(\delta)$ is a nonlinear function of the intertube spacing δ. In the pre-buckling situation, $\delta(x,y) \equiv \delta_0$; therefore, we can obtain the following:

$$p_{10}^{\ V}(x,y) = g(\delta_0) \tag{17}$$

where $p_{10}^{\ V}(x,y)$ is the vdW force of the outer tube prior to the buckling, which is defined as the value of g at the initial interlayer spacing δ_0 between the inner and the outer tubes. After buckling, the pressure caused by the vdW force at any point (x,y) on the outer tube in the present linearized analysis is obtained from the following [4]:

$$p_1^{\ V}(x,y) = p_{10}^{\ V} + C^*[W_2(x,y) - W_1(x,y)] \tag{18a}$$

Substituting Eq. (18a) into Eq. (15) yields the Eq.presented as:

$$p_2^{\ V}(x,y) = -\frac{r_1}{r_2}\left\{p_{10}^{\ V} + C^*[W_2(x,y) - W_1(x,y)]\right\} \ , \tag{18b}$$

where C^* is a constant [2,4] defined as follows:

$$C^* = \left.\frac{dg}{d\delta}\right|_{\delta=\delta_0} \tag{18c}$$

The interaction pressure due to the elastic medium on the outer tube $p_1^{\ W-P}$ can be given in the following form:

$$p_1^{\ W-P} = p_{10}^{\ W-P} - F \ , \tag{19}$$

where $p_{10}^{\ W-P}$ is the interaction pressure prior to buckling. Substituting Eq. (12a) into Eq. (19) results in the following expression:

$$p_1^{\ W-P} = p_{10}^{\ W-P} - KW_1(x,y) + G\nabla^2 W_1(x,y) \ , \tag{20}$$

(after buckling $K > 0$ and $G > 0$)

where K is the spring constant of the Winkler type, and G denotes the shear constant of the Pasternak type.

Using Eq. (11), the governing equations of the equilibrium for the outer and the inner tubes are written as follows:

$$
\begin{aligned}
&D\nabla^8 W_1 + \frac{1-\mu^2}{r_1^2} C W_{1,xxxx} \\
&-\left[1-(e_0a)^2\nabla^2\right]\nabla^4\left[2N_{xy01}W_{1,xy} + N_{y01}W_{1,yy}\right] \\
&= \left[1-(e_0a)^2\nabla^2\right]\nabla^4 p_1 \\
&D\nabla^8 W_2 + \frac{1-\mu^2}{r_2^2} C W_{2,xxxx} \\
&-\left[1-(e_0a)^2\nabla^2\right]\nabla^4\left[2N_{xy02}W_{2,xy} + N_{y02}W_{2,yy}\right] \\
&= \left[1-(e_0a)^2\nabla^2\right]\nabla^4 p_2
\end{aligned}
\tag{21}
$$

Moreover, $N_{y01} = -N_{y02}$ and $N_{xy01} = N_{xy02} = N_{xy0}$ [4].

The total torque applied to the two ends of a DWCNT is as follows:

$$
T = T_1 + T_2 , \tag{22a}
$$

where, T_1 and T_2 are the torque applied to the outer and the inner tubes, respectively. Assuming that the shear membrane forces in each layer of a DWCNT are identical, the following relations can be written:

$$
N_{xy01} = N_{xy02} = \frac{T_1}{2\pi r_1^2} = \frac{T_2}{2\pi r_2^2} \tag{22b}
$$

Substituting Eqs. (18a), (18b), and (20) into Eqs. (13) and (14), and by applying Eq. (21), the governing equations of the equilibrium for a DWCNT are obtained as follows:

$$
\begin{aligned}
&D\nabla^8 W_1 + \frac{1-\mu^2}{r_1^2} C W_{1,xxxx} \\
&-2N_{xy0}\left[1-(e_0a)^2\nabla^2\right]\nabla^4 W_{1,xy} \\
&+N_{y02}\left[1-(e_0a)^2\nabla^2\right]\nabla^4 W_{1,yy} = \\
&C^*\left[1-(e_0a)^2\nabla^2\right]\nabla^4 W_2 \\
&-\left(C^*+K\right)\left[1-(e_0a)^2\nabla^2\right]\nabla^4 W_1 + G\left[1-(e_0a)^2\nabla^2\right]\nabla^6 W_1
\end{aligned}
\tag{23a}
$$

$$
\begin{aligned}
&D\nabla^8 W_2 + \frac{1-\mu^2}{r_2^2} C W_{2,xxxx} \\
&-2N_{xy0}\left[1-(e_0a)^2\nabla^2\right]\nabla^4 W_{2,xy} \\
&-N_{y02}\left[1-(e_0a)^2\nabla^2\right]\nabla^4 W_{2,yy} = \\
&C^*\frac{r_1}{r_2}\left[1-(e_0a)^2\nabla^2\right]\left(\nabla^4 W_1 - \nabla^4 W_2\right)
\end{aligned}
\tag{23b}
$$

In Eqs. (23a) and (23b), the effects of the Winkler and Pasternak foundations, the small scale, and the pre-buckling vdW force are considered.

The buckling modes are assumed as

$$
\begin{aligned}
W_1(x,y) &= A_1 \sin\left(\frac{m\pi}{L}x - \frac{ny}{r_1}\right) \\
W_2(x,y) &= A_2 \sin\left(\frac{m\pi}{L}x - \frac{ny}{r_2}\right)
\end{aligned}
, \tag{24}
$$

where $A_i (i=1,2)$ are real constants, L is the length of DWCNT, and m and n are the axial half and the circumferential wave numbers, respectively.

Eq. (24) satisfies the simple supported boundary conditions and the governing equations of the equilibrium.

Substituting Eq. (24) into Eqs. (23a) and (23b) yields the following equations:

$$
\begin{aligned}
L_{11}W_1 + L_{12}W_2 &= 0 \\
L_{21}W_1 + L_{22}W_2 &= 0
\end{aligned}
\tag{25a}
$$

Eq. (25a) can be written in the following matrix form:

$$
\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}
\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix}
= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
\tag{25b}
$$

In order to obtain a non-trivial solution for Eq. (25a), it is necessary to set the determinant of the coefficient matrix in Eq. (25b) to zero:

$$
L_{11}L_{22} - L_{12}L_{21} = 0 \tag{26}
$$

Solving Eq. (26) yields the critical buckling load of the DWCNT for different wave numbers m and n. The effect of small scale, i.e. the e_0a term in the constitutive equation (Eq. (1) was first introduced by Eringen [15]. Therefore, the critical buckling load, which is obtained using the small-scale effect, is called the nonlocal critical buckling load. Moreover, if the effect of small scale is set to zero ($e_0a = 0$), the resulting critical buckling load will be called the local critical buckling load.

where, L_{11}, L_{12}, L_{21}, and L_{22} in Eq. (26) are given by the following:

$$
\begin{aligned}
L_{11} = {} & D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]^4 + \frac{1-\mu^2}{r_1^2} C\left(\frac{m\pi}{L}\right)^4 \\
& - \left\{2N_{xy0}\left(\frac{m\pi}{L}\right)\left(\frac{n}{r_1}\right) + N_{y02}\left(\frac{n}{r_1}\right)^2\right\} \\
& \times \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]^2 \left\{1 + (e_0a)^2\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]\right\} \\
& + \left\{(C^*+K) + G\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]\right\} \\
& \times \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]^2 \left\{1 + (e_0a)^2\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]\right\}
\end{aligned}
\tag{27a}
$$

$$
\begin{aligned}
L_{12} = {} & -C^*\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_2}\right)^2\right]^2 \\
& \times \left\{1 + (e_0a)^2\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_2}\right)^2\right]\right\}
\end{aligned}
\tag{27b}
$$

$$
\begin{aligned}
L_{21} = {} & -C^*\frac{r_1}{r_2}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]^2 \\
& \times \left\{1 + (e_0a)^2\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{r_1}\right)^2\right]\right\}
\end{aligned}
\tag{27c}
$$

$$L_{22} = D\left[(\frac{m\pi}{L})^2 + (\frac{n}{r_2})^2\right]^4 + \frac{1-\mu^2}{r_2^2}C(\frac{m\pi}{L})^4$$
$$+\left\{-2N_{xy0}(\frac{m\pi}{L})(\frac{n}{r_2}) + C^*\frac{r_1}{r_2} + N_{y02}(\frac{n}{r_2})^2\right\} \tag{27d}$$
$$\times\left[(\frac{m\pi}{L})^2 + (\frac{n}{r_2})^2\right]^2 \left\{1+(e_0a)^2\left[(\frac{m\pi}{L})^2 + (\frac{n}{r_2})^2\right]\right\}$$

3. Simplification and discussion

In this section, the critical buckling load of a DWCNT subjected to torsional load was investigated. For long radius of CNTs, the difference between the inner and the outer radii should be much smaller than the radius of the DWCNT: it can be assumed that the following is true [4]:

$$r_1 \approx r_2 = r \tag{28}$$

Substituting Eq. (28) into Eq. (26), an explicit expression for the nonlocal critical shear membrane force can be given as follows:

$$N_{xy0} = \frac{1}{2(\frac{m\pi}{L})(\frac{n}{r})} \times \frac{1}{1+(e_0a)^2 \times A}\left\{D A^2\right\}$$
$$+\frac{1}{2(\frac{m\pi}{L})(\frac{n}{r})} \times \frac{1}{1+(e_0a)^2 \times A}\left\{\frac{1-\mu^2}{r^2}C(\frac{m\pi}{L})^4 \times \frac{1}{A^2}\right\}$$
$$+\frac{1}{2(\frac{m\pi}{L})(\frac{n}{r})}\left\{C^* + \frac{K}{2} + \frac{G}{2}A\right\} \tag{29}$$
$$-\frac{1}{2(\frac{m\pi}{L})(\frac{n}{r})}\left\{\sqrt{C^{*2} + \left[-\frac{K}{2} - \frac{G}{2}A + N_{y02}(\frac{n}{r})^2\right]^2}\right\}$$

where $A = \left[(\frac{m\pi}{L})^2 + (\frac{n}{r})^2\right]$.

By definition, $\bar{m} = \frac{m\pi\, r}{L}$ Eq. (29) can be written as follows:

$$N_{xy0} = B\left[\frac{D}{r^2}\frac{(\bar{m}^2+n^2)^2}{2\bar{m}n}\right]$$
$$+B\left[(1-\mu^2)C\frac{\bar{m}^3}{2n(\bar{m}^2+n^2)^2}\right]$$
$$+\frac{r^2}{2\bar{m}n}\left\{C^* + \frac{K}{2} + \frac{G}{2}\frac{(\bar{m}^2+n^2)}{r^2}\right\} \tag{30}$$
$$-\frac{r^2}{2\bar{m}n}\sqrt{C^{*2} + \left[-\frac{K}{2} - \frac{G}{2}\frac{(\bar{m}^2+n^2)}{r^2} + N_{y02}(\frac{n}{r})^2\right]^2}$$

where $B = \frac{1}{1+(e_0a)^2\frac{(\bar{m}^2+n^2)}{r^2}}$

The above equation is the most complete formula for the torsional buckling of a DWCNT because both the small-scale effect and the effect of the surrounding elastic medium, including the Winkler and Pasternak models have been considered in its development.

3.1 In the absence of the vdW forces with $C^* = 0$

In this section, the critical buckling load of a DWCNT subjected to torsional loading in the absence of vdW forces ($C^* = 0$) was investigated. In this state, the inner tube behaves as an SWCNT. Taking K and $G > 0$ after buckling, the nonlocal critical buckling load can be derived using Eq. (30):

$$N_{xy0} = B\left[\frac{D}{r^2}\frac{(\bar{m}^2+n^2)^2}{2\bar{m}n} + (1-\mu^2)C\frac{\bar{m}^3}{2n(\bar{m}^2+n^2)^2}\right]$$
$$+\frac{r^2}{2\bar{m}n}\left\{K + G\frac{(\bar{m}^2+n^2)}{r^2}\right\} \tag{31}$$

It can be seen clearly in Eq. (31) that the surrounding elastic medium increases the nonlocal critical buckling load of the outer tube. Moreover, for $K > 0$ and $G = 0$ the following relation holds using Eq. (31):

$$N_{xy0} = B\left[\frac{D}{r^2}\frac{(\bar{m}^2+n^2)^2}{2\bar{m}n} + (1-\mu^2)C\frac{\bar{m}^3}{2n(\bar{m}^2+n^2)^2}\right]$$
$$+\frac{r^2}{2\bar{m}n}K \tag{32}$$

Comparing Eqs. (31) and (32), one can find that the surrounding elastic medium, considering the shear constant of the Pasternak type, increases the nonlocal critical buckling load of the outer tube with respect to that which is equal to zero.

3.2 Effect of the surrounding elastic medium

In this section, the effect of the surrounding elastic medium on the critical buckling load was investigated. The following inequality universally holds:

$$\left[\frac{K}{2} + \frac{G}{2}\frac{(\bar{m}^2+n^2)}{r^2} + \sqrt{C^{*2} + N^2_{y02}(\frac{n}{r})^4}\right]$$
$$> \sqrt{C^{*2} + \left[-\frac{K}{2} - \frac{G}{2}\frac{(\bar{m}^2+n^2)}{r^2} + N_{y02}(\frac{n}{r})^2\right]^2} \tag{33}$$

The above inequality can be rewritten as follows:

$$\left\{C^* + \frac{K}{2} + \frac{G}{2}\frac{(\bar{m}^2+n^2)}{r^2}\right.$$
$$\left.-\sqrt{C^{*2} + \left[-\frac{K}{2} - \frac{G}{2}\frac{(\bar{m}^2+n^2)}{r^2} + N_{y02}(\frac{n}{r})^2\right]^2}\right\} \tag{34}$$
$$> C^* - \sqrt{C^{*2} + N^2_{y02}(\frac{n}{r})^4}$$

The nonlocal critical buckling load of a DWCNT in the absence of the surrounding elastic medium with $K=0$ and $G=0$ yields the following:

$$N_{xy0-nonlocal} = B\left[\frac{D}{r^2}\frac{(\bar{m}^2+n^2)^2}{2\bar{m}n} + (1-\mu^2)C\frac{\bar{m}^3}{2n(\bar{m}^2+n^2)^2}\right] + \frac{r^2}{2\bar{m}n}\left\{C^* - \sqrt{C^{*2}+N^2_{y02}(\frac{n}{r})^4}\right\} \quad (35)$$

Using Eq. (34), it can be shown that the nonlocal critical buckling load given by Eq. (30) is higher than that predicted by Eq. (35). It can thus concluded that the surrounding elastic medium increases the nonlocal critical buckling load of a DWCNT.

3.3 In the absence of the surrounding elastic medium and the vdW forces with K = 0 , G = 0 , and C^* = 0

In this section, the nonlocal shear membrane force in the absence of the vdW force and the surrounding elastic medium was studied, with the inner tube behaving as a SWCNT. The nonlocal critical buckling load is given by the following:

$$N_{xy0-nonlocal} = B\left[\frac{D}{r^2}\frac{(\bar{m}^2+n^2)^2}{2\bar{m}n} + (1-\mu^2)C\frac{\bar{m}^3}{2n(\bar{m}^2+n^2)^2}\right] \quad (36)$$

The small-scale effect is clearly seen in the above equation, which describes the nonlocal critical buckling load for a SWCNT.

Ignoring the small-scale effect in Eq. (36) yields the following local (classical) critical buckling load:

$$N_{xy0-local} = \frac{D}{r^2}\frac{(\bar{m}^2+n^2)^2}{2\bar{m}n} + (1-\mu^2)C\frac{\bar{m}^3}{2n(\bar{m}^2+n^2)^2} \quad (37)$$

For the long cylinders, the shell buckles in two circumferential waves, i.e., the smallest values of N_{xy0} correspond to $n=2$.

$$N_{xy0-local} = \frac{D}{r^2}\frac{(\bar{m}^2+4)^2}{4\bar{m}} + (1-\mu^2)C\frac{\bar{m}^3}{4(\bar{m}^2+4)^2} \quad (38)$$

For long cylinders, $\bar{m}^2 << 4$. Thus, approximately, the local buckling load is written as follows:

$$N_{xy0-local} = \frac{4D}{\bar{m}r^2} + (1-\mu^2)C\frac{\bar{m}^3}{64} \quad (39)$$

Setting $dN_{xy0}/d\bar{m}=0$ yields the following:

$$\bar{m}^4 = \frac{64}{9(1-\mu^2)}(\frac{t}{r})^2 \quad (40)$$

Substituting Eq. (40) into Eq. (39), the minimum local critical buckling load was obtained as follows:

$$N_{xy0-local-cr} = 0.272\frac{Et}{(1-\mu^2)^{3/4}}(\frac{t}{r})^{3/2} \quad (41)$$

For $\tau_{cr} = N_{xy0-local-cr}/t$, the following relation was obtained:

$$\tau_{cr} = 0.272\frac{E}{(1-\mu^2)^{3/4}}(\frac{t}{r})^{3/2} \quad (42)$$

For $\mu=0.3$, Eq. (42) becomes the following:

$$\tau_{cr} = 0.292E(\frac{t}{r})^{3/2} \quad (43)$$

4. General discussion

The effects of the surrounding elastic medium and the small scale of the nonlocal critical buckling load under torsional loading for a DWCNT were investigated. Using Eq. (12b), one can find the value of G as follows:

$$\frac{\bar{G}}{\bar{\mu}} = \frac{G}{Kr^2} \quad (44)$$

Therefore, the following holds:

$$G = Kr^2\frac{\bar{G}}{\bar{\mu}} \quad (45)$$

The mechanical properties of DWCNT, the surrounding elastic medium, and the small-scale effect can be considered as follows [3, 23]:

$$r_1 = 1.02nm, r_2 = 0.68nm, E = 1Tpa$$
$$t = 0.34nm, \mu = 0.3, C^* = 9.91866693\times10^{19}\ N/m^3 \quad (46)$$

The e_0 was chosen to be equal to 0.39 [15], and the value of parameter a, the length of a C-C bond, was chosen as $0.142nm$; therefore, e_0a is equal to 0.05538. According to [22], $\bar{G}$ and $\bar{\mu}$ should be of the same order; hence, substituting Eq. (46) into Eq. (12b) yields the following:

$$\bar{\mu} = 0.0055648, \quad (47)$$

and

$$\bar{G} = 0, 0.0055, 0.0085 \quad (48)$$

Substituting Eqs. (47) and (48) into Eq. (45), the following parameter was obtained:

$$G = 0, 2.071273, 3.201058\ N/m \quad (49)$$

Fig. 2 shows the local critical buckling load under torsional loading versus the axial half sine wave number for the circumferential sine wave numbers of 1, 2, 3 and 4. It can be observed from the results that the obtained critical buckling loads for the values of the axial half and the circumferential sine wave numbers are equal to 4 and 2, respectively.

Meanwhile, Fig. 3 illustrates the local and the nonlocal crit-

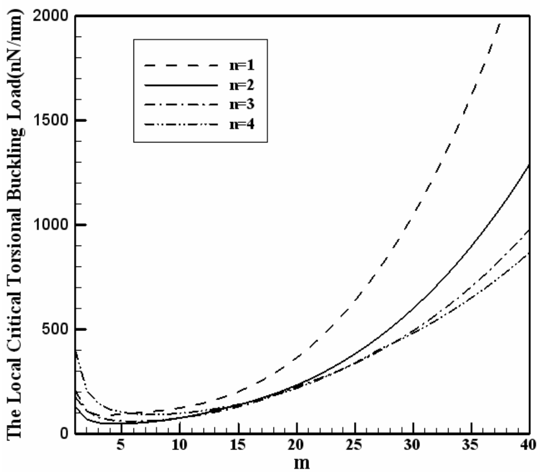

Fig. 2. The Local critical buckling load versus the axial half sine wave number for $L/r_1 = 10$, $K = 8.9995035\times10^{17}\ N/m^3$, $N_{y02} = 2\,N/m$, $G = 2.071273\,N/m$, $e_0a = 0$.

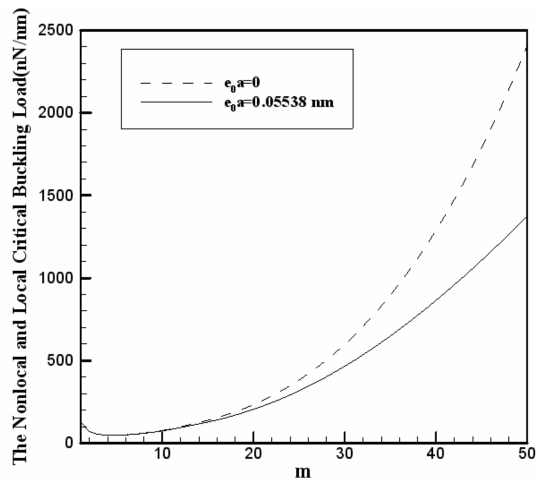

Fig. 3. The small-scale effect on the torsional buckling load for $n = 2$, $L/r_1 = 10$, $K = 8.9995035\times10^{17}\ N/m^3$, $N_{y02} = 2\,N/m$, $G = 2.071273\,N/m$.

ical buckling loads under torsional loading versus the axial half sine wave number for the circumferential sine wave number of 2 ($n = 2$). It can be seen from this figure that the nonlocal critical buckling load is lower than the local critical buckling load. The difference between the local and the nonlocal critical buckling loads increases along with the increase in the half axial sine wave number.

Fig. 4 describes the effect of the surrounding elastic medium such as the spring constant of the Winkler type on the nonlocal critical buckling load for a DWCNT. It can be observed that the nonlocal critical buckling load increases with increasing the spring constant of the Winkler type (K).

Fig. 5 shows the effect of the surrounding elastic medium, such as the shear constant of the Pasternak type on the nonlocal critical buckling load for a DWCNT. It is concluded that the nonlocal critical torsional buckling load increases by increasing the shear constant of the Pasternak type (G). More-

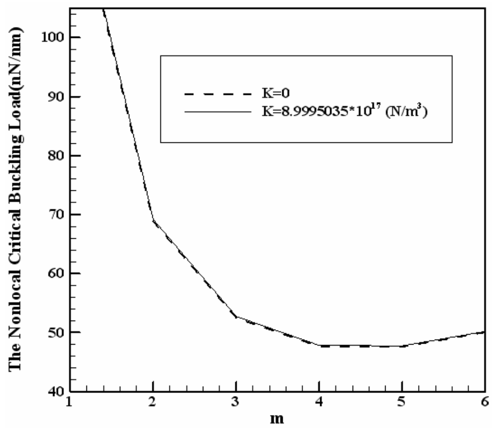

Fig. 4. The effect of the spring constant of the Winkler type on the nonlocal critical buckling load for $n = 2$, $L/r_1 = 10$, $N_{y02} = 2\,N/m$, $G = 2.071273\,N/m$, $e_0a = 0.05538\,nm$.

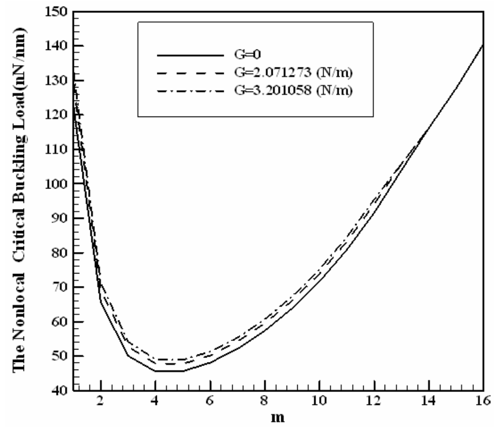

Fig. 5. The effect of the shear constant of the Pasternak type on the nonlocal critical buckling load for $n = 2$, $L/r_1 = 10$, $K = 8.9995035\times10^{17}\ N/m^3$, $N_{y02} = 2\,N/m$, $e_0a = 0.05538\,nm$.

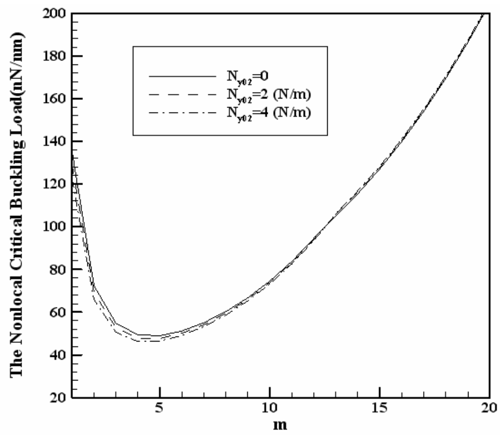

Fig. 6. The effect of the vdW force of prebuckling on the nonlocal critical buckling load for $n = 2$, $L/r_1 = 10$, $K = 8.9995035\times10^{17}$, N/m^3, $G = 2.071273\,N/m$, $e_0a = 0.05538\,nm$.

over, the difference between considering and not considering the Pasternak foundation is significant; as a result, the effect of this parameter cannot be neglected in the formulations.

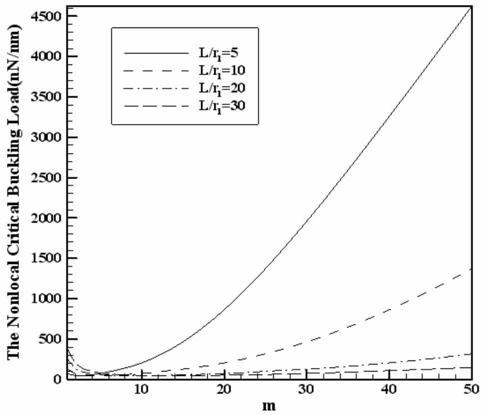

Fig. 7. The nonlocal critical buckling load versus the axial half sine wave number for aspect ratios of 5, 10, 20, and 30 and for $n = 2$, , $K = 8.9995035\times10^{17}\ N/m^3$, $N_{y02} = 2\ N/m$, $G = 2.071273\ N/m$, $e_0a = 0.05538\ nm$.

Meanwhile, Fig. 6 depicts the effect of the vdW force prior to the buckling on the nonlocal critical buckling load. It can be seen that the nonlocal critical buckling load decreases with increasing pre-buckling vdW (N_{y02}).

Fig. 7 shows the nonlocal critical buckling load under the torsional loading versus the axial half sine wave number for aspect ratios (L/r_1) of 5, 10, 20, and 30. It can be seen that the nonlocal critical buckling load decreases with increasing aspect ratios. Moreover, the nonlocal critical buckling load is nearly constant for high aspect ratios.

5. Conclusions

Using the continuum cylindrical shell model, the small-scale effect on the critical torsional buckling load of a DWCNT embedded on Winkler and Pasternak foundations was investigated. The effects of the surrounding elastic medium, such as the spring constant of the Winkler type and the shear constant of the Pasternak type and vdW forces between the inner and the outer nanotubes, were taken into consideration. The obtained results showed that the local critical buckling load was higher than the nonlocal critical buckling load. On the other hand, it was observed that the critical buckling load increased with increasing axial half sine wave number (m), and that the difference between the nonlocal and the local buckling loads increased at high axial half sine wave number. As a result, the small-scale effect on the torsional buckling load should not be neglected. Generally, the nonlocal critical bucking load increases in the presence of the surrounding elastic medium, including the Winkler and Pasternak foundations. In addition, the effect of the spring constant of the Winkler type (K) becomes less than the shear constant of the Pasternak type (G). In this study, it was observed that the shear constant of the Pasternak type increased the nonlocal critical torsional buckling load, while the difference between the presence and the absence of the shear constant of Pasternak type becomes large. According to the results, the nonlocal critical torsional buckling load decreases with increasing the vdW force of pre-buckling. Finally, a simplified analysis was also presented to estimate the nonlocal critical torque for torsional buckling of a DWCNT.

The results of this research can be used for studying the torsional buckling behavior of a DWCNT embedded on Winkler and Pasternak foundations.

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Appendix

Eqs. (9a), (9b), and (9c) were decoupled in the following manner:

1. By differentiating Eq. (9b) with respect to the independent variables x and y, the Eq. (9b) can be written as follows:

$$\frac{1+\mu}{2}u_{1,xyxy}+\frac{1-\mu}{2}v_{1,xxxy}+v_{1,yyxy}+\frac{w_{1,yxy}}{r}=0 \tag{A1}$$

2. The following Eq.was obtained by differentiating Eq. (9a) with respect to the independent variable x twice:

$$u_{1,xxxx}+\frac{1-\mu}{2}u_{1,yyxx}+\frac{1+\mu}{2}v_{1,xyxx}+\mu\frac{w_{1,xxx}}{r}=0 \tag{A2}$$

3. The governing Eq.of the equilibrium can be written as the following by differentiating Eq. (9a) with respect to the independent variable y twice:

$$u_{1,xxyy}+\frac{1-\mu}{2}u_{1,yyyy}+\frac{1+\mu}{2}v_{1,xyyy}+\mu\frac{w_{1,xyy}}{r}=0 \tag{A3}$$

4. Using Eqs. (A2) and (A3), the following equations were obtained:

$$v_{1,xyxx}=\frac{-2}{1+\mu}\left[u_{1,xxxx}+\frac{1-\mu}{2}u_{1,yyxx}+\mu\frac{w_{1,xxx}}{r}\right], \text{ and} \tag{A4}$$

$$v_{1,xyyy}=\frac{-2}{1+\mu}\left[u_{1,xxyy}+\frac{1-\mu}{2}u_{1,yyyy}+\mu\frac{w_{1,xyy}}{r}\right] \tag{A5}$$

5. Substituting Eqs. (A4) and (A5) into Eq. (A1) and simplifying the results yield the following equation:

$$\nabla^4 u_1=-\frac{\mu}{r}w_{1,xxx}+\frac{1}{r}w_{1,xyy} \tag{A6}$$

6. Similarly, by differentiating Eq. (9a) with respect to the independent variables x and y, this can be written as the following:

$$u_{1,xxxy}+\frac{1-\mu}{2}u_{1,yyxy}+\frac{1+\mu}{2}v_{1,xyxy}+\mu\frac{w_{1,xxy}}{r}=0 \tag{A7}$$

7. Differentiating Eq. (9b) with respect to the independent variable x twice yields the following equation:

$$\frac{1+\mu}{2}u_{1,xyxx}+\frac{1-\mu}{2}v_{1,xxxx}+v_{1,yyxx}+\frac{w_{1,yxx}}{r}=0 \tag{A8}$$

8. Differentiating Eq. (9b) with respect to the independent variable y twice yields the following equation:

$$\frac{1+\mu}{2}u_{1,xyyy}+\frac{1-\mu}{2}v_{1,xxyy}+v_{1,yyyy}+\frac{w_{1,yyy}}{r}=0 \tag{A9}$$

9. Rearranging Eqs. (A8) and (A9), the following equations were obtained:

$$u_{1,xyxx}=\frac{-2}{1+\mu}\left[\frac{1-\mu}{2}v_{1,xxxx}+v_{1,yyxx}+\frac{w_{1,yxx}}{r}\right], \text{ and} \tag{A10}$$

$$u_{1,xyyy}=\frac{-2}{1+\mu}\left[\frac{1-\mu}{2}v_{1,xxyy}+v_{1,yyyy}+\frac{w_{1,yyy}}{r}\right] \tag{A11}$$

10. Substituting Eqs. (A10) and (A11) into Eq. (A7) and simplifying the results yield the following equation:

$$\nabla^4 v_1=-\frac{2+\mu}{r}w_{1,xxy}-\frac{1}{r}w_{1,yyy} \tag{A12}$$

11. By differentiating operator ∇^4 of Eq. (9c), the following Eq.was obtained:

$$\begin{aligned}&D\nabla^8 w_1+\frac{1}{r^2}C(r\nabla^4 v_{1,y}+\nabla^4 w_1+\mu r\nabla^4 u_{1,x})\\&-\left[1-(e_0a)^2\nabla^2\right]\nabla^4\left[N_{x0}w_{1,xx}+2N_{xy0}w_{1,xy}+N_{y0}w_{1,yy}\right]\\&=\left[1-(e_0a)^2\nabla^2\right]\nabla^4 p(x,y)\end{aligned} \tag{A13}$$

where $\nabla^4()=\frac{\partial^4()}{\partial x^4}+2\frac{\partial^4()}{\partial x^2\partial y^2}+\frac{\partial^4()}{\partial y^4}$.

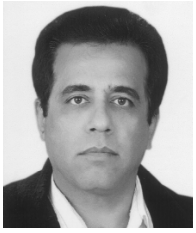
12. Substituting Eqs. (A6) and (A12) into Eq. (A13), the gov-

erning Eq. of the equilibrium can be written as follows:

$$D\nabla^8 w_1 + \frac{1-\mu^2}{r^2} C w_{1,xxxx} - \left[1-(e_0 a)^2 \nabla^2\right] \nabla^4 \left[N_{x0} w_{1,xx} + 2N_{xy0} w_{1,xy} + N_{y0} w_{1,yy}\right] = \left[1-(e_0 a)^2 \nabla^2\right] \nabla^4 p \tag{A14}$$

Mehdi Mohammadimehr received his B.Sc. degree from the University of Kashan in Kashan, Iran, in 2002. He then received his M.Sc. degree from Shahid Bahonar University of Kerman in Kerman, Iran, in 2004. He is currently a Ph.D student at Shahid Bahonar University of Kerman. His research interests are buckling and vibration analyses of carbon nanotubes and functionally graded materials (FGMs).

Ali Reza Saidi received his B.Sc. degree from Amirkabir University of Technology in Tehran, Iran, in 1993. He received his M.Sc. and Ph.D degrees from Sharif University of Technology in Tehran, Iran, in 1996 and 2003, respectively. Dr Ali Reza Saidi is currently an Associate Professor at the Department of Mechanical Engineering, Shahid Bahonar University of Kerman in Kerman, Iran. His research interests include elasticity, plasticity, nonlinear continuum mechanics, large deformation analysis of solids, nanomechanics, composite materials, functionally graded materials (FGMs), and plates and shells.



Ali Ghorbanpour Arani received his B.Sc. degree from Sharif University of Technology in Tehran, Iran, in 1988. He then received his M.Sc. degree from Amirkabir University of Technology in Tehran, Iran, in 1991 and his Ph.D degree from the Esfahan University of Technology in Esfahan, Iran, in 2001. Dr Ali Ghorbanpour Arani is currently a Professor at the Mechanical Engineering Department of University of Kashan in Kashan, Iran. His current research interests are stress analyses, stability and vibration of nanotubes, and FGMs.